

Parafermions in the τ_2 model

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Abstract. It has been shown recently by Baxter that the $\tau_2(t_q)$ model with open boundary conditions can be solved by the “parafermionic” method of Fendley. In Baxter’s paper there are several conjectures, which were formulated based on numerical short-chain calculations. Here we present the proof of two of them.

1. Introduction

Parastatistics, more general than Bose or Fermi statistics, has been advocated first by Green [1] in 1953, albeit that some form of generalized statistics was already present implicitly in the Bethe Ansatz paper [2] of 1931. In 1967, the mathematician Morris introduced a generalization of the classical Clifford algebra [3, 4, 5],[‡] which can be used to describe ‘cyclic’ parafermions with statistics very different from Green’s. It may also have to be noted that the special case of the Morris algebra with only two generators is known as the Weyl pair [8]§. In 1980 Fradkin and Kadanoff [10] proposed that clock-type models in two dimensions are an ideal laboratory to study such parafermions, generated via short-distance expansion of the product of order and disorder variables, thus generalizing ideas of Kadanoff and Ceva for the Ising model [11].

Many papers have followed since [10] appeared, including many papers on the N -state chiral Potts model. We should particularly mention two papers by Fendley [12, 13] on the chiral Potts quantum spin chain and its specialization for $N = 2$, the quantum Ising chain. In these two papers, the parafermion operators—introduced in section 3.3 of [12]—are almost identical in form with the E operators of Morris [3, 4, 5]. However, unlike the Ising case, commuting the Hamiltonian of a chain of length L with a linear combination of $2L$ of such parafermions does not give rise to another linear combination of such parafermions [12]. Fendley [14] considered next the ‘simple’ Hamiltonian introduced by Baxter [15] and he constructed NL cyclic raising operators

[‡] Yamazaki [6], Popovici and Ghéorghe [7] wrote about this algebra without giving an explicit representation.

[§] Sylvester introduced this already in 1883 in his paper [9] on quaternions, nonions, sedenions, etc.

(called shift operators by him) [14], allowing him to obtain the complete spectrum of the Hamiltonian.

Baxter constructed his simple Hamiltonian as a special limit [16] of what is now known as the τ_2 model, which he obtained specializing the parameters in the integrable chiral Potts model [17]. The existence of this τ_2 model was already implicit in two papers [18, 19] by Krichever, who derived the genus of the underlying curve as $(N - 1)^2$, but only gave explicit details for case $N = 2$, in which he derived the full free-fermion model as a descendant. Korepanov continued Krichever's program, and discovered the τ_2 model for $N = 3$ in 1986 and for general N in 1987 [20], explicitly giving the curve with genus $(N - 1)^2$. Unfortunately, Korepanov's work was only known to very few people in Russia until about 1993.

The τ_2 model became really important through the work of Bazhanov and Stroganov [21], who obtained the chiral Potts model as a descendant of the six-vertex model. They showed that the R -matrix (20) in [17], which is made up of four chiral Potts model weights, intertwines two τ_2 R -matrices. More precisely they found a sequence of four Yang-Baxter equations, in which each of the three rapidity lines could either have a six-vertex model rapidity or two chiral Potts model rapidities. The name τ_2 came up when the authors of [22] decided to introduce the τ_n model, whose R -matrix intertwines a spin- S highest-weight representation and a cyclic representation while $n = 2S + 1$; more simply said, one rapidity line belongs to the spin- S generalization of the six-vertex model and the other carries two chiral Potts model rapidities. Ever since the two papers [21, 22] most calculations in the integrable chiral Potts model [17, 23] have used the τ_2 model at one point or another, exploiting the commutation properties of the transfer matrices of the two models as implied by the Yang-Baxter equations.

Recently Baxter [24] generalized Fendley's method to an inhomogeneous $\tau_2(t)$ model with open boundary conditions. Its transfer matrix $\boldsymbol{\tau}_2(t)$ is known to be a polynomial in t of degree L , $\boldsymbol{\tau}_2(t) = \sum_{n=0}^L (-\omega t)^n \mathbf{A}_n$ and the Hamiltonian is given as $\mathcal{H} = -\mathbf{A}_1/\mathbf{A}_0$. All operators here are explicitly defined in the next section. In (B4.1) and (B4.2)⌋, Baxter defined iteratively⌋

$$\boldsymbol{\Gamma}_0 = \mathbf{Z}_1^{-1}, \quad \boldsymbol{\Gamma}_{j+1} = (\omega^{-1} - 1)^{-1}(\mathcal{H}\boldsymbol{\Gamma}_j - \boldsymbol{\Gamma}_j\mathcal{H}). \quad (1)$$

Based on numerical evidence, Baxter found (B4.3), i.e. that

$$s_0\boldsymbol{\Gamma}_{NL+j} + s_1\boldsymbol{\Gamma}_{NL-N+j} + \cdots + s_L\boldsymbol{\Gamma}_j = 0, \quad \text{for } j \geq 0, \quad (2)$$

holds, so that at most NL of the $\boldsymbol{\Gamma}_j$ are linearly independent, allowing us to truncate the infinite sequence of them. Furthermore, Baxter showed that there exists a linear transformation to transform the NL operators $\boldsymbol{\Gamma}_j$ to cyclic raising operators $\widehat{\boldsymbol{\Gamma}}_j$. He

⌋ Equations in [24] are denoted here by prefacing B to the equation number.

⌋ We note that (33) and (34) in [14], (4) and (1) in [15], (8.20) and (8.21) in [16], and (B1.2) in [24] are consistent with one another identifying $\sigma_m \equiv \mathbf{Z}_m$ and $\tau_m \equiv \mathbf{X}_m$. However, Hamiltonian (B1.5) is spatially reflected compared to the other papers. Therefore, we must choose (B4.1) and not $\boldsymbol{\Gamma}_0 = \mathbf{Z}_1$ as would agree with $\psi_1 = \sigma_1$ chosen in subsection 5.1 of [14].

showed that these operators are to satisfy (B5.2) and (B5.4), as conjectured based on numerical evidence for spin chains of length up to 6.

Finally, to obtain the spectrum of $\tau_2(t)$, he defined in (B4.7)

$$\boldsymbol{\mu}_j \equiv \boldsymbol{\Gamma}_j \boldsymbol{\tau}_2(t) - \boldsymbol{\tau}_2(t) \boldsymbol{\Gamma}_j, \quad \boldsymbol{\nu}_j \equiv \omega \boldsymbol{\Gamma}_j \boldsymbol{\tau}_2(t) - \boldsymbol{\tau}_2(t) \boldsymbol{\Gamma}_j, \quad (3)$$

and observed numerically that $t\boldsymbol{\nu}_j = \boldsymbol{\mu}_{j-1}$ in (B4.8). One expects some relation of this type replacing the $\boldsymbol{\Gamma}_j$ in the $\boldsymbol{\nu}_j$ by $\boldsymbol{\Gamma}_{j-1}$ upon using (1). The definition of $\boldsymbol{\nu}_j$ has an ω -commutator and this must be seen as a consequence of the denominator $(\omega^{-1} - 1)$ in (1). That this has to be so can be most easily seen expanding the special case $t\boldsymbol{\nu}_1 = \boldsymbol{\mu}_0$ to linear order in t using $\boldsymbol{\tau}_2(t) = \mathbf{1} + \omega t \mathcal{H} + O(t^2)$.

In this paper we shall present proofs of conjectures (B4.3) and (B4.8). We shall also simplify (B5.4) and present explicit forms for the operators. We are using the method of commuting transfer matrices within the Yang–Baxter approach, which is very different from the method of Fendley [14], who uses the method of iteratively constructing commuting local Hamiltonians. Nevertheless, his generating function $T(t)$ in his equations (48) and (50) is simply related to our $\boldsymbol{\tau}_2(t)$ in the special limit (B3.25), as we shall show in Appendix B.

2. Transfer matrix and Hamiltonian

The transfer matrix of the generalized τ_2 model [25] can be written as a product of interaction-round-a-face weights, as was done in [24] based on (14) and Figure 4 of [25]. Alternatively, it can also be written as a product of vertex-model \mathcal{L} -matrices as indicated in Figure 5 of [25]. In fact, equation (20) in [25] gives the \mathcal{L} -matrix acting on vector \mathbf{g}_J , so that we can express the τ_2 transfer matrix as the 2×2 trace

$$\tau_2(t) = \text{trace} \left(\prod_{j=0}^L \mathcal{L}_j \right). \quad (4)$$

From Appendix A we obtain⁺

$$\begin{aligned} \mathcal{L}_j(m_{j-1}, m_j; \sigma_j, \sigma'_j) &= \mathcal{L}_j(m_{j-1}, m_j)_{\sigma_j, \sigma'_j} \\ &= \omega^{m_j \sigma'_j - m_{j-1} \sigma_j} (-\omega t_q)^{\sigma_j - \sigma'_j - m_{j-1}} F_{2j-2}(\sigma_j - \sigma'_j | m_{j-1}) F_{2j-1}(\sigma_j - \sigma'_j | m_j), \end{aligned} \quad (5)$$

using (A.12), (A.14) and (A.15). Also, we must identify $F_{2j-2}(n|m) = F_{p_{2j-2}, q}(n, m)$ and $F_{2j-1}(n|m) = F_{p_{2j-1}, q}(n, m)$ when comparing with (B2.2). Rewriting the \mathcal{L}_j as 2-by-2 matrices with N -by- N matrix elements, we thus find

$$\begin{aligned} \mathcal{L}_j(0, 0) &= b_{2j-2} b_{2j-1} - \omega t_q d_{2j-2} d_{2j-1} \mathbf{X}_j, \\ \mathcal{L}_j(0, 1) &= (-\omega t_q) \mathbf{Z}_j (b_{2j-2} c_{2j-1} - d_{2j-2} a_{2j-1} \mathbf{X}_j), \\ \mathcal{L}_j(1, 0) &= \mathbf{Z}_j^{-1} (c_{2j-2} b_{2j-1} - \omega a_{2j-2} d_{2j-1} \mathbf{X}_j), \\ \mathcal{L}_j(1, 1) &= \omega a_{2j-2} a_{2j-1} \mathbf{X}_j - \omega t_q c_{2j-2} c_{2j-1}, \end{aligned} \quad (6)$$

⁺ Compare (5) with the action on vector \mathbf{g}_J in (20) of [25], identifying $m_{j-1} = m$, $m_j = m'$, $\sigma_j = a$, $\sigma'_j = d$. The difference is a factor $(-\omega t_q)^{m-m'}$ corresponding to a simple gauge transformation.

where

$$\begin{aligned} [\mathbf{Z}_j]_{\sigma, \sigma'} &= \omega^{\sigma_j} \prod_{k=0}^L \delta(\sigma_k, \sigma'_k), \quad [\mathbf{X}_j]_{\sigma, \sigma'} = \delta(\sigma_j, \sigma'_j + 1) \prod_{k \neq j} \delta(\sigma_k, \sigma'_k), \\ \mathbf{Z}_j \mathbf{X}_j &= \omega \mathbf{X}_j \mathbf{Z}_j. \end{aligned} \quad (7)$$

Particularly, for $c_{2L} \equiv c_{-2} = c_{-1} = 0$, $a_{-1} = d_{-1} = 0$ and $b_{-1} = b_{-2} = 1$, in agreement with (B3.1), (B3.4) and (B3.6), we find

$$\mathcal{L}_0 = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}. \quad (8)$$

Let

$$\prod_{j=1}^L \mathcal{L}_j = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t) \\ \mathbf{C}(t) & \mathbf{D}(t) \end{bmatrix}, \quad (9)$$

then from (4), (8), and (9), we find

$$\tau_2(t) = \mathbf{A}(t) = \sum_{\ell=0}^L \mathbf{A}_\ell (-\omega t)^\ell, \quad (10)$$

where the \mathbf{A}_ℓ are operators commuting with one another. Indeed $[\tau_2(t), \tau_2(t')] = 0$ as follows from Yang–Baxter equation (A.24), which is valid for all inhomogeneous choices of the rapidities $p_j = \{a_j, b_j, c_j, d_j\}$ [25]. Next we rewrite (6) as

$$\mathcal{L}_j = \mathcal{L}_j^+ - \omega t \mathcal{L}_j^-, \quad (11)$$

where the \mathcal{L}_j^+ and \mathcal{L}_j^- are both triangular,

$$\mathcal{L}_j^+ = \begin{bmatrix} \alpha_j^+ & 0 \\ \beta_j^+ & \gamma_j^+ \end{bmatrix}, \quad \mathcal{L}_j^- = \begin{bmatrix} \alpha_j^- & \beta_j^- \\ 0 & \gamma_j^- \end{bmatrix}, \quad (12)$$

and respectively given by the constant terms or the linear terms in (6). Consequently, we find*

$$\mathbf{A}_0 = \prod_{j=1}^L \alpha_j^+ = \left[\prod_{j=0}^{2L-1} b_j \right] \mathbf{1}, \quad \mathbf{A}_L = \prod_{j=1}^L \alpha_j^- = \left[\prod_{j=0}^{2L-1} d_j \right] \mathbf{X}_1 \cdots \mathbf{X}_L, \quad (13)$$

with $\mathbf{A}_0 = A_0 \mathbf{1}$ and $\alpha_j^+ = \alpha_j^+ \mathbf{1}$ proportional to the unit operator, and the Hamiltonian

$$\mathcal{H} = -\frac{\mathbf{A}_1}{A_0} = -\sum_{j=1}^L \left[\frac{\alpha_j^-}{\alpha_j^+} + \frac{\beta_j^-}{\alpha_j^+} \sum_{m=j+1}^L \left(\prod_{\ell=j+1}^{m-1} \frac{\gamma_\ell^+}{\alpha_\ell^+} \right) \frac{\beta_m^+}{\alpha_m^+} \right], \quad (14)$$

can be easily shown to be identical to (B3.22). It should be noted that this Hamiltonian is not the one of the integrable chiral Potts chain [23] as studied by Fendley [12, 13], but it reduces in the special limit (B3.25) to the one he studied in [14], see appendix B.

Since, from (B4.1),

$$\Gamma_0 = \mathbf{Z}_1^{-1}, \quad (15)$$

* We do not set $b_j \equiv 1$ as done in [24], so that we can treat the superintegrable case later.

we may split the product in (9) into two parts

$$\prod_{j=1}^L \mathcal{L}_j = \mathcal{L}_1 \prod_{j=2}^L \mathcal{L}_j, \quad (16)$$

and rewrite the second part as

$$\prod_{j=2}^L \mathcal{L}_j = \begin{bmatrix} \mathbf{A}^{2,L}(t) & \mathbf{B}^{2,L}(t) \\ \mathbf{C}^{2,L}(t) & \mathbf{D}^{2,L}(t) \end{bmatrix}, \quad (17)$$

which makes explicit that it is a 2×2 matrix with operator entries. It follows that

$$\boldsymbol{\tau}_2(t) = \mathbf{A}(t) = (\boldsymbol{\alpha}_1^+ - \omega t \boldsymbol{\alpha}_1^-) \mathbf{A}^{2,L}(t) - \omega t \boldsymbol{\beta}_1^- \mathbf{C}^{2,L}(t), \quad (18)$$

where

$$\boldsymbol{\alpha}_1^+ = b_0 b_1 \mathbf{1}, \quad \boldsymbol{\alpha}_1^- = d_0 d_1 \mathbf{X}_1, \quad \boldsymbol{\beta}_1^- = \mathbf{Z}_1 (b_0 c_1 - d_0 a_1 \mathbf{X}_1). \quad (19)$$

Expanding

$$\mathbf{A}^{2,L}(t) = \sum_{\ell=0}^{L-1} \hat{\mathbf{A}}_\ell (-\omega t)^\ell, \quad \mathbf{C}^{2,L}(t) = \sum_{\ell=0}^{L-1} \hat{\mathbf{C}}_\ell (-\omega t)^\ell, \quad (20)$$

and substituting this and (10) into (18), we can relate the coefficients as

$$\mathbf{A}_\ell = \boldsymbol{\alpha}_1^+ \hat{\mathbf{A}}_\ell + \boldsymbol{\alpha}_1^- \hat{\mathbf{A}}_{\ell-1} + \boldsymbol{\beta}_1^- \hat{\mathbf{C}}_{\ell-1}. \quad (21)$$

Particularly, the Hamiltonian (14) can be rewritten as

$$-\mathbf{A}_0 \mathcal{H} = \mathbf{A}_1 = \boldsymbol{\alpha}_1^+ \hat{\mathbf{A}}_1 + \boldsymbol{\alpha}_1^- \hat{\mathbf{A}}_0 + \boldsymbol{\beta}_1^- \hat{\mathbf{C}}_0. \quad (22)$$

Obviously, as $\hat{\mathbf{A}}_\ell$ and $\hat{\mathbf{C}}_\ell$ are operators acting on sites from 2 to L , they commute with $\boldsymbol{\Gamma}_0$, $\boldsymbol{\alpha}_1^\pm$ and $\boldsymbol{\beta}_1^-$. Using this, the iterative definition (B4.2), i.e.

$$\boldsymbol{\Gamma}_j \mathbf{A}_1 - \mathbf{A}_1 \boldsymbol{\Gamma}_j = (\omega^{-1} - 1) \mathbf{A}_0 \boldsymbol{\Gamma}_{j+1}, \quad (23)$$

the third equation (7) rewritten as $\mathbf{X}_1 \boldsymbol{\Gamma}_0 = \omega \boldsymbol{\Gamma}_0 \mathbf{X}_1$, and (22), we find

$$\mathbf{A}_0 \boldsymbol{\Gamma}_1 = \omega (\boldsymbol{\Gamma}_0 \boldsymbol{\alpha}_1^- \hat{\mathbf{A}}_0 - d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_0). \quad (24)$$

We shall first prove (B4.8), which is the easiest.

3. Proof of (B4.8)

From the definitions of $\boldsymbol{\mu}_j$ and $\boldsymbol{\nu}_j$ in (3), cf. (B4.7), and the definition of the Gamma matrices in (1), we find

$$\mathcal{H} \boldsymbol{\mu}_j - \boldsymbol{\mu}_j \mathcal{H} = \boldsymbol{\mu}_{j+1} (\omega^{-1} - 1), \quad \mathcal{H} \boldsymbol{\nu}_j - \boldsymbol{\nu}_j \mathcal{H} = \boldsymbol{\nu}_{j+1} (\omega^{-1} - 1). \quad (25)$$

Thus if we can prove $t \boldsymbol{\nu}_1 = \boldsymbol{\mu}_0$, then by repeated application of (25) on both sides, we can obtain $t \boldsymbol{\nu}_{j+1} = \boldsymbol{\mu}_j$. Using the expansion in (10) and the definitions (3), we find

$$\begin{aligned} t \boldsymbol{\nu}_1 &= \sum_{\ell=1}^L (-\omega t)^\ell (\omega^{-1} \mathbf{A}_{\ell-1} \boldsymbol{\Gamma}_1 - \boldsymbol{\Gamma}_1 \mathbf{A}_{\ell-1}), \\ \boldsymbol{\mu}_0 &= \sum_{\ell=1}^L (-\omega t)^\ell (\boldsymbol{\Gamma}_0 \mathbf{A}_\ell - \mathbf{A}_\ell \boldsymbol{\Gamma}_0). \end{aligned} \quad (26)$$

If we can prove that the coefficients of t^ℓ are identical, then the identity is proven. It is easily seen from (23) that this equality holds for $\ell = 1$. From (21) and (19), we find

$$\Gamma_0 \mathbf{A}_\ell - \mathbf{A}_\ell \Gamma_0 = (1 - \omega)(\Gamma_0 \alpha_1^- \hat{\mathbf{A}}_{\ell-1} - d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_{\ell-1}), \quad (27)$$

and from (21) and (24), we obtain

$$\begin{aligned} & \omega^{-1} \mathbf{A}_{\ell-1} \Gamma_1 - \Gamma_1 \mathbf{A}_{\ell-1} \\ &= A_0^{-1} \left[(1 - \omega) \Gamma_0 (\alpha_1^+ \hat{\mathbf{A}}_0) \alpha_1^- \hat{\mathbf{A}}_{\ell-1} + (\beta_1^- \Gamma_0 \alpha_1^- - \omega \Gamma_0 \alpha_1^- \beta_1^-) \hat{\mathbf{A}}_0 \hat{\mathbf{C}}_{\ell-2} \right. \\ & \quad \left. + \alpha_1^+ d_0 a_1 \mathbf{X}_1 (\omega \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_{\ell-1} - \hat{\mathbf{A}}_{\ell-1} \hat{\mathbf{C}}_0) + \alpha_1^- d_0 a_1 \mathbf{X}_1 (\omega \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_{\ell-2} - \hat{\mathbf{A}}_{\ell-2} \hat{\mathbf{C}}_0) \right], \end{aligned} \quad (28)$$

where the relations

$$\beta_1^- \mathbf{X}_1 = \omega \mathbf{X}_1 \beta_1^-, \quad \alpha_1^- \Gamma_0 = \omega \Gamma_0 \alpha_1^-, \quad (29)$$

and (A.25) have also been used.

From the Yang–Baxter equations \sharp we obtain the relation

$$\omega^{-1} (1 - x/y) \mathbf{A}(y) \mathbf{C}(x) + (1 - \omega^{-1}) \mathbf{C}(y) \mathbf{A}(x) = (1 - \omega^{-1} x/y) \mathbf{C}(x) \mathbf{A}(y). \quad (30)$$

By equating the coefficients, we find

$$\hat{\mathbf{A}}_\ell \hat{\mathbf{C}}_0 - \omega \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_\ell = (1 - \omega) \hat{\mathbf{C}}_\ell \hat{\mathbf{A}}_0 = (1 - \omega) \hat{\mathbf{A}}_0 \hat{\mathbf{C}}_\ell, \quad (31)$$

and

$$\hat{\mathbf{A}}_1 \hat{\mathbf{C}}_\ell - \hat{\mathbf{C}}_\ell \hat{\mathbf{A}}_1 = (1 - \omega) (\hat{\mathbf{C}}_{\ell+1} \hat{\mathbf{A}}_0 - \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_{\ell+1}) \quad (32)$$

$$\begin{aligned} &= \hat{\mathbf{A}}_{\ell+1} \hat{\mathbf{C}}_0 - \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_{\ell+1} \\ &= (1 - \omega^{-1}) (\hat{\mathbf{A}}_{\ell+1} \hat{\mathbf{C}}_0 - \hat{\mathbf{A}}_0 \hat{\mathbf{C}}_{\ell+1}). \end{aligned} \quad (33)$$

To go from (32) to (33) via the indicated intermediate step requires two applications of (31). From (13) and (19), we have

$$(\alpha_1^+ \hat{\mathbf{A}}_0) = \mathbf{A}_0, \quad \beta_1^- \Gamma_0 \alpha_1^- - \omega \Gamma_0 \alpha_1^- \beta_1^- = (1 - \omega) d_0 a_1 \alpha_1^- \mathbf{X}_1. \quad (34)$$

There are four terms within the square brackets of (28). Using (34) for the first and second terms and (31) for the third and fourth terms, one can show that the right-hand sides of (27) and (28) are equal, so that

$$\omega^{-1} \mathbf{A}_{\ell-1} \Gamma_1 - \Gamma_1 \mathbf{A}_{\ell-1} = \Gamma_0 \mathbf{A}_\ell - \mathbf{A}_\ell \Gamma_0 \quad (35)$$

for all ℓ . Thus we have proven the identity (B4.8) in [24].

4. Proof of (B4.3)

4.1. Explicit form of Γ_j

In (24), Γ_1 is explicitly given. We shall prove by induction that for $\ell \geq 1$

$$\Gamma_\ell = \omega^\ell \sum_{m=0}^{\ell-1} (-1)^m \mathbf{R}_m \mathbf{q}_{\ell-1-m} = \omega \sum_{m=0}^{\ell-1} (-1)^m \mathbf{q}_{\ell-1-m} \mathbf{R}_m, \quad (36)$$

\sharp Details will be discussed in the Appendix, as there are rather subtle differences depending on the various conventions.

where

$$\mathbf{R}_m \equiv \Gamma_0 \alpha_1^- \hat{\mathbf{A}}_m - d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_m, \quad (37)$$

in which the hatted operators do not commute with the \mathbf{A}_m , but commute with α_1^\pm , β_1^- and Γ_0 , while the \mathbf{q}_ℓ are operators which can be obtained iteratively by the relations

$$\mathbf{q}_0 = \frac{1}{A_0}, \quad \mathbf{q}_\ell = \sum_{n=1}^{\ell} (-1)^{n+1} \frac{A_n}{A_0} \mathbf{q}_{\ell-n}. \quad (38)$$

Since the \mathbf{q}_ℓ are expressed in terms of the \mathbf{A}_n , they commute with all \mathbf{A}_n . The second equality in (36) is needed only for the next section.

Comparing (36) with (24), we find it gives the right result for Γ_1 . Now we assume (24) holds for Γ_ℓ , and prove it is also correct for $\Gamma_{\ell+1}$. Using (23) and (36), we find

$$(1 - \omega^{-1}) \mathbf{A}_0 \Gamma_{\ell+1} = \mathbf{A}_1 \Gamma_\ell - \Gamma_\ell \mathbf{A}_1 = \omega^\ell \sum_{m=0}^{\ell-1} (-1)^m (\mathbf{A}_1 \mathbf{R}_m - \mathbf{R}_m \mathbf{A}_1) \mathbf{q}_{\ell-1-m}, \quad (39)$$

$$= \omega \sum_{m=0}^{\ell-1} (-1)^m \mathbf{q}_{\ell-1-m} (\mathbf{A}_1 \mathbf{R}_m - \mathbf{R}_m \mathbf{A}_1). \quad (40)$$

Using (37) we split the commutator $\mathbf{A}_1 \mathbf{R}_m - \mathbf{R}_m \mathbf{A}_1$ into two parts $\mathbf{I}_1 - \mathbf{I}_2$, with

$$\mathbf{I}_1 = \mathbf{A}_1 \Gamma_0 \alpha_1^- \hat{\mathbf{A}}_m - \Gamma_0 \alpha_1^- \hat{\mathbf{A}}_m \mathbf{A}_1, \quad \mathbf{I}_2 = d_0 a_1 (\mathbf{A}_1 \mathbf{X}_1 \hat{\mathbf{C}}_m - \mathbf{X}_1 \hat{\mathbf{C}}_m \mathbf{A}_1). \quad (41)$$

After substituting (22) into (41), we use the commutation relations (29) and (A.25) and the fact that the hatted operators commute with all operators on site 1 and find

$$\mathbf{I}_1 = (\omega - 1) \Gamma_0 (\alpha_1^-)^2 \hat{\mathbf{A}}_0 \hat{\mathbf{A}}_m + \alpha_1^- [\beta_1^- \Gamma_0 \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m - \omega^{-1} \Gamma_0 \beta_1^- \hat{\mathbf{A}}_m \hat{\mathbf{C}}_0] \quad (42)$$

$$= (\omega - 1) \Gamma_0 (\alpha_1^-)^2 \hat{\mathbf{A}}_0 \hat{\mathbf{A}}_m + \alpha_1^- [(\beta_1^- \Gamma_0 - \Gamma_0 \beta_1^-) \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m + \Gamma_0 \beta_1^- (\hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m - \omega^{-1} \hat{\mathbf{A}}_m \hat{\mathbf{C}}_0)]. \quad (43)$$

Next we use (31) and combining (19) and (15) we may write

$$\beta_1^- \Gamma_0 - \Gamma_0 \beta_1^- = -(\omega - 1) d_0 a_1 \mathbf{X}_1, \quad (44)$$

so that

$$\begin{aligned} \mathbf{I}_1 &= (1 - \omega^{-1}) \left[\omega \Gamma_0 (\alpha_1^-)^2 \hat{\mathbf{A}}_0 \hat{\mathbf{A}}_m + \alpha_1^- (\Gamma_0 \beta_1^- \hat{\mathbf{C}}_m \hat{\mathbf{A}}_0 - \omega d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m) \right] \\ &= (1 - \omega^{-1}) \left[\omega \Gamma_0 \alpha_1^- (\alpha_1^- \hat{\mathbf{A}}_m + \beta_1^- \hat{\mathbf{C}}_m) \hat{\mathbf{A}}_0 - \omega \alpha_1^- d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m \right], \end{aligned} \quad (45)$$

from which, using (21) and (34), we obtain

$$\mathbf{I}_1 = (1 - \omega^{-1}) \omega \left[\Gamma_0 \alpha_1^- (\hat{\mathbf{A}}_0 \mathbf{A}_{m+1} - \hat{\mathbf{A}}_{m+1} \mathbf{A}_0) - \alpha_1^- d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m \right]. \quad (46)$$

Similarly, we find, using (22), (29) and (A.25),

$$\mathbf{I}_2 = d_0 a_1 \left[\alpha_1^+ \mathbf{X}_1 (\hat{\mathbf{A}}_1 \hat{\mathbf{C}}_m - \hat{\mathbf{C}}_m \hat{\mathbf{A}}_1) + (\omega - 1) \mathbf{X}_1 \beta_1^- \hat{\mathbf{C}}_0 \hat{\mathbf{C}}_m \right], \quad (47)$$

which, upon using first (32) and in the next step (34) and (21), becomes

$$\begin{aligned} \mathbf{I}_2 &= - (1 - \omega^{-1}) \omega d_0 a_1 \mathbf{X}_1 \left[\alpha_1^+ \hat{\mathbf{C}}_{m+1} \hat{\mathbf{A}}_0 - \hat{\mathbf{C}}_0 (\alpha_1^+ \hat{\mathbf{A}}_{m+1} + \beta_1^- \hat{\mathbf{C}}_m) \right] \\ &= - (1 - \omega^{-1}) \omega d_0 a_1 \mathbf{X}_1 (\hat{\mathbf{C}}_{m+1} \mathbf{A}_0 - \hat{\mathbf{C}}_0 \mathbf{A}_{m+1} + \alpha_1^- \hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m). \end{aligned} \quad (48)$$

Combining (46) and (48), we find that the last terms in the two equations cancel out. The definition of \mathbf{R}_m in (37) is then used to write

$$[\mathbf{A}_1, \mathbf{R}_m] = \mathbf{I}_1 - \mathbf{I}_2 = (1 - \omega^{-1})\omega(\mathbf{R}_0 \mathbf{A}_{m+1} - \mathbf{R}_{m+1} A_0). \quad (49)$$

Substituting (49) into (39), we find

$$\Gamma_{\ell+1} = \omega^{\ell+1} \left[\mathbf{R}_0 \sum_{m=0}^{\ell-1} (-1)^m \frac{\mathbf{A}_{m+1}}{A_0} \mathbf{q}_{\ell-1-m} - \sum_{m=0}^{\ell-1} (-1)^m \mathbf{R}_{m+1} \mathbf{q}_{\ell-1-m} \right]. \quad (50)$$

Noticing from (38) that the coefficient of \mathbf{R}_0 is \mathbf{q}_ℓ and replacing m by $m-1$ in the second sum, we find $\Gamma_{\ell+1}$ is also of the form (36), thus completing the proof of the first equality in (36).

Alternatively, we may rewrite (42) as

$$\begin{aligned} \mathbf{I}_1 &= (\omega - 1) \Gamma_0 (\boldsymbol{\alpha}_1^-)^2 \hat{\mathbf{A}}_0 \hat{\mathbf{A}}_m \\ &\quad + \boldsymbol{\alpha}_1^- [\beta_1^- \Gamma_0 (\hat{\mathbf{C}}_0 \hat{\mathbf{A}}_m - \omega^{-1} \hat{\mathbf{A}}_m \hat{\mathbf{C}}_0) + \omega^{-1} (\beta_1^- \Gamma_0 - \Gamma_0 \beta_1^-) \hat{\mathbf{A}}_m \hat{\mathbf{C}}_0]. \end{aligned} \quad (51)$$

After again using (44) and (31) and performing a few commutations with the help of (29) and (A.25), we can apply (21) to arrive at the alternative form

$$\mathbf{I}_1 = (1 - \omega^{-1}) \left[\mathbf{A}_{m+1} \Gamma_0 \boldsymbol{\alpha}_1^- \hat{\mathbf{A}}_0 - \mathbf{A}_0 \Gamma_0 \boldsymbol{\alpha}_1^- \hat{\mathbf{A}}_{m+1} - \boldsymbol{\alpha}_1^- d_0 a_1 \mathbf{X}_1 \hat{\mathbf{A}}_m \hat{\mathbf{C}}_0 \right]. \quad (52)$$

Next, similar to what we did in deriving (48), we now use commutation relation (33) followed by (29) and (21) to rewrite (47) as

$$\mathbf{I}_2 = (1 - \omega^{-1}) \left[\mathbf{A}_{m+1} d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_0 - \mathbf{A}_0 d_0 a_1 \mathbf{X}_1 \hat{\mathbf{C}}_{m+1} - \boldsymbol{\alpha}_1^- d_0 a_1 \mathbf{X}_1 \hat{\mathbf{A}}_m \hat{\mathbf{C}}_0 \right], \quad (53)$$

so that

$$[\mathbf{A}_1, \mathbf{R}_m] = \mathbf{I}_1 - \mathbf{I}_2 = (1 - \omega^{-1}) (\mathbf{A}_{m+1} \mathbf{R}_0 - A_0 \mathbf{R}_{m+1}). \quad (54)$$

Consequently, we find (40) becomes

$$\Gamma_{\ell+1} = \omega \left[\left(\sum_{m=1}^{\ell} (-1)^{m+1} \mathbf{q}_{\ell-m} \frac{\mathbf{A}_m}{A_0} \right) \mathbf{R}_0 + \sum_{m=1}^{\ell} (-1)^m \mathbf{q}_{\ell-m} \mathbf{R}_m \right]. \quad (55)$$

Again we use (38) to show that the second equality in (36) also holds for $\ell+1$.

4.2. Proof of (B4.3)

We first rewrite (38) as

$$\sum_{n=0}^{\ell} (-1)^n \mathbf{A}_n \mathbf{q}_{\ell-n} = \delta_{\ell,0} \mathbf{1}. \quad (56)$$

Because $\mathbf{A}_n = 0$ for $n > L$ and $\mathbf{q}_\ell = 0$ for $-N < \ell < 0$,^{††} the upper limit of the summation can be replaced by L or larger. It is easily seen from (10) that

$$\begin{aligned} \prod_{n=0}^{N-1} \tau_2(\omega^n t) &= \sum_{\ell=0}^L s_\ell t^{N\ell} \mathbf{1} = \prod_{n=0}^{N-1} \left[\sum_{\ell_n=0}^L \mathbf{A}_{\ell_n} (-\omega^{n+1} t)^{\ell_n} \right] \\ &= \sum_{m=0}^{NL} (-t)^m \sum_{\ell_1+\dots+\ell_N=m} \mathbf{A}_{\ell_1} \mathbf{A}_{\ell_2} \dots \mathbf{A}_{\ell_N} \omega^{\ell_1+2\ell_2+\dots+N\ell_N}. \end{aligned} \quad (57)$$

^{††} Compare eqs. (50) of [27] and (71) of [28] and nearby text.

As it is obvious that

$$\sum_{\ell_1+\dots+\ell_N=m} \cdots \sum \mathbf{A}_{\ell_1} \mathbf{A}_{\ell_2} \cdots \mathbf{A}_{\ell_N} \omega^{\ell_1+2\ell_2+\dots+N\ell_N} = 0 \quad \text{for } m \neq jN, \quad (58)$$

we recover (B3.14) with

$$s_j \mathbf{1} = (-1)^{jN} \sum_{\ell_1+\dots+\ell_N=jN} \cdots \sum \mathbf{A}_{\ell_1} \mathbf{A}_{\ell_2} \cdots \mathbf{A}_{\ell_N} \omega^{\ell_1+2\ell_2+\dots+N\ell_N}. \quad (59)$$

Now consider the sum

$$\mathbf{K} = \sum_{\ell=0}^L s_\ell \Gamma_{NL-\ell N}. \quad (60)$$

Substituting (59) into it, and using (58), we rewrite it as

$$\begin{aligned} \mathbf{K} &= \sum_{m=0}^{NL} \Gamma_{NL-m} (-1)^m \sum_{\ell_1+\dots+\ell_N=m} \cdots \sum \mathbf{A}_{\ell_1} \mathbf{A}_{\ell_2} \cdots \mathbf{A}_{\ell_N} \omega^{\ell_1+2\ell_2+\dots+N\ell_N} \\ &= \sum_{\ell_1=0}^L \cdots \sum_{\ell_N=0}^L \Gamma_{NL-\ell_1-\dots-\ell_N} (-1)^{\ell_1+\dots+\ell_N} \mathbf{A}_{\ell_1} \mathbf{A}_{\ell_2} \cdots \mathbf{A}_{\ell_N} \omega^{\ell_1+2\ell_2+\dots+N\ell_N}. \end{aligned} \quad (61)$$

Since (36) is not valid for Γ_0 , it cannot be used when $\ell_1 = \dots = \ell_N = L$ in the above N -fold sum. Setting this term, which is easily simplified, apart and denoting the remaining $(L+1)^N - 1$ terms by putting primes on the sums, we split \mathbf{K} into two parts. Next we substitute (36) into the remaining sum part, after changing the upper limit of the summation of (36) to $L-1$. Because $\hat{\mathbf{A}}_m = \hat{\mathbf{C}}_m = 0$ for $m \geq L$ and $\mathbf{q}_\ell = 0$ for $-L < \ell < 0$, the two choices of the upper limits for m are equivalent. Thus we arrive at

$$\begin{aligned} \mathbf{K} &= \Gamma_0 (-1)^{NL} (\mathbf{A}_L)^N \omega^{\frac{1}{2}N(N+1)L} + \\ &\sum_{\ell_1=0}^L \cdots \sum_{\ell_N=0}^L \sum_{m=0}^{L-1} \mathbf{R}_m \mathbf{q}_{NL-m-\ell_1-\dots-\ell_N-1} (-1)^{m+\ell_1+\dots+\ell_N} \mathbf{A}_{\ell_1} \mathbf{A}_{\ell_2} \cdots \mathbf{A}_{\ell_N} \omega^{\ell_2+\dots+(N-1)\ell_N}. \end{aligned} \quad (62)$$

The summation over ℓ_1 can be carried out using (56) resulting in

$$\begin{aligned} \mathbf{K} &= \Gamma_0 (-1)^{NL} (\mathbf{A}_L)^N \omega^{\frac{1}{2}N(N+1)L} + \\ &\sum_{\ell_2=0}^L \cdots \sum_{\ell_N=0}^L \sum_{m=0}^{L-1} \mathbf{R}_m \delta_{NL-1, m+\ell_2+\dots+\ell_N} (-1)^{m+\ell_2+\dots+\ell_N} \mathbf{A}_{\ell_2} \cdots \mathbf{A}_{\ell_N} \omega^{\ell_2+\dots+(N-1)\ell_N}. \end{aligned} \quad (63)$$

There is only way for $m + \ell_2 + \dots + \ell_N = NL - 1$ to hold, namely $m = L - 1$ and $\ell_2 = \dots = \ell_N = L$. From (11) and (12) we can easily see that $\hat{\mathbf{C}}_{L-1} = 0$, and from (13) we find

$$\hat{\mathbf{A}}_{L-1} = \prod_{j=2}^L \alpha_j^-, \quad \text{so that} \quad \mathbf{R}_{L-1} = \Gamma_0 \mathbf{A}_L, \quad (64)$$

as seen from (37) and (19). Consequently, (63) becomes

$$\mathbf{K} = \sum_{\ell=0}^L s_\ell \Gamma_{NL-\ell N} = \Gamma_0 (-1)^{NL+NL+L} (\mathbf{A}_L)^N + \Gamma_0 (-1)^{NL-1+NL-L} (\mathbf{A}_L)^N = 0. \quad (65)$$

This proves (B4.3). In fact, for $j > 0$ it is more straightforward to prove (2) by simply substituting (36) into the sum and then use (56).

5. Explicit form of $\widehat{\Gamma}_j$

5.1. Eigenvectors of \mathbf{H}

It is worth noting that even though the s_j given in (59) are expressed in terms of operators, they are scalars as seen from (B3.13) and (B3.14). Thus the elements of the \mathbf{H} -matrix given in (B4.10) and (B4.11) are also scalars. They can be rewritten as

$$\begin{aligned} h_{ij} &= \delta_{i+1,j} \quad \text{for } 0 \leq i \leq NL-2, 0 \leq j \leq NL-1; \\ h_{NL-1,jN} &= -s_{L-j}/s_0, \quad h_{NL-1,m} = 0 \quad \text{for } m \neq Nj. \end{aligned} \quad (66)$$

The eigenvalues of \mathbf{H} are given by Baxter in the text below (B4.19) as $\lambda_i = r_k \omega^p$, ($0 \leq p \leq N-1$), and the r_k^N are the roots of the polynomial

$$s_0 \lambda^{NL} + s_1 \lambda^{NL-N} + \cdots + s_{L-1} \lambda^N + s_L = s_0 \prod_{j=1}^L (\lambda^N - r_j^N). \quad (67)$$

Let $\mathbf{V}^{(i)} = (V_0^{(i)}, V_1^{(i)}, \dots, V_{NL-1}^{(i)})$ denote the eigenvector whose eigenvalue is λ_i . Then

$$(\mathbf{H}\mathbf{V}^{(i)})_j = V_{j+1}^{(i)} = \lambda_i V_j^{(i)}, \quad \text{for } 0 \leq j \leq NL-2, \quad (68)$$

so that

$$V_j^{(i)} = \lambda_i^j, \quad \text{for } 0 \leq j \leq NL-1, \quad (69)$$

if we choose the normalization $V_0^{(i)} = 1$. Consider now the last row of \mathbf{H} given explicitly in (B4.10) or also (66). It follows that

$$(\mathbf{H}\mathbf{V}^{(i)})_{NL-1} = (-s_L V_0^{(i)} - s_{L-1} V_N^{(i)} - \cdots - s_1 V_{NL-N}^{(i)})/s_0 = \lambda_i V_{NL-1}^{(i)}, \quad (70)$$

which is seen from (69) to be

$$-s_L - s_{L-1} \lambda_i^N - \cdots - s_1 \lambda_i^{NL-N} = s_0 \lambda_i^{NL}, \quad (71)$$

consistent with (69) for $j = NL-1$. Since $\lambda_i^N = r_i^N$ are the roots of the above polynomial (67), we find that the $\mathbf{V}^{(i)}$ with elements given by (69) are indeed the eigenvectors of \mathbf{H} . Obviously matrix \mathbf{P} diagonalizing \mathbf{H} is a Vandermonde matrix, and the elements of its inverse $(P^{-1})_{ik}$ are the coefficients of the polynomials $f_i(z)$ given by

$$f_i(z) = \prod_{j=1, j \neq i}^{NL} \frac{z - \lambda_j}{\lambda_i - \lambda_j} = \sum_{k=0}^{NL-1} (P^{-1})_{ik} z^k, \quad \text{satisfying } f_i(\lambda_j) = \delta_{ij}. \quad (72)$$

This is essentially Prony's 1795 result [26, 27].

5.2. Alternative form for \mathbf{q}_ℓ

Let $\mathbf{Q}(t)$ be a polynomial given by

$$\mathbf{Q}(t) = \sum_{\ell=0}^{\infty} \mathbf{q}_\ell (\omega t)^\ell. \quad (73)$$

Because of (56), we find

$$\mathbf{A}(t)\mathbf{Q}(t) = \sum_{m=0}^{\infty} (\omega t)^m \sum_{\ell=0}^m (-1)^\ell \mathbf{A}_\ell \mathbf{q}_{m-\ell} = \sum_{m=0}^{\infty} (\omega t)^m \delta_{m,0} \mathbf{1} = \mathbf{1}. \quad (74)$$

Consequently, we have $\mathbf{Q}(t) = \mathbf{A}(t)^{-1}$. If we rewrite $\mathbf{A}(t)$ as

$$\mathbf{A}(t) = \sum_{\ell=0}^L \mathbf{A}_\ell (-\omega t)^\ell = A_0 \prod_{\ell=1}^L (1 - \omega t \mathbf{u}_\ell), \quad (75)$$

where the \mathbf{u}_ℓ are commuting operators, as the set of $\mathbf{A}(t)$ for varying t forms a commuting family. Since the eigenvalues of $\boldsymbol{\tau}_2(t)$ are given by Baxter in (B3.19) as

$$A_0 \prod_{\ell=1}^L (1 - r_\ell \omega^{n_\ell+1} t), \quad 0 \leq n_\ell \leq N-1, \quad (76)$$

the eigenvalues of \mathbf{u}_ℓ are $r_\ell \omega^{n_\ell}$. We find

$$\mathbf{Q}(t) = \frac{1}{\mathbf{A}(t)} = \sum_{\ell=1}^L \frac{\boldsymbol{\Omega}_\ell}{1 - \omega t \mathbf{u}_\ell} = \sum_{\ell=1}^L \boldsymbol{\Omega}_\ell \sum_{m=0}^{\infty} (\omega t \mathbf{u}_\ell)^m = \sum_{m=0}^{\infty} (\omega t)^m \sum_{\ell=1}^L \boldsymbol{\Omega}_\ell \mathbf{u}_\ell^m, \quad (77)$$

where $\boldsymbol{\Omega}_\ell$ can be easily found by the residue theorem. This means that \mathbf{q}_m has an alternative expression,

$$\mathbf{q}_m = \sum_{\ell=1}^L \boldsymbol{\Omega}_\ell \mathbf{u}_\ell^m, \quad \boldsymbol{\Omega}_\ell = \left[A_0 \prod_{n=1, n \neq \ell}^L (1 - \mathbf{u}_n / \mathbf{u}_\ell) \right]^{-1}. \quad (78)$$

It can be shown as was done in our previous work [27, 28] that this expression for \mathbf{q}_m is valid for all $m > -L$, and is identically 0 when $-L < m < 0$, see particularly eqs. (50) of [27] and (71) of [28] and nearby text.

5.3. Explicit Form of $\hat{\Gamma}_j$

From the first equality in (36) and definition (B4.17), we find

$$\hat{\Gamma}_i = \sum_{j=0}^{NL-1} P_{ij}^{-1} \Gamma_j = \sum_{m=0}^{L-1} \mathbf{R}_m (-1)^m \sum_{j=0}^{NL-1} P_{ij}^{-1} \omega^j \mathbf{q}_{j-1-m}, \quad (79)$$

in which the elements of \mathbf{P}^{-1} are given in (72), and $\mathbf{q}_{j-1-m} = 0$ for $j-1 < m < L$ as was said below (78). We follow the convention of Baxter to denote the i th eigenvalues of \mathbf{H} by $\lambda_{p,k} = r_k \omega^p$, i.e. identifying $i = (p, k)$, and substitute (78) into the above equation to obtain

$$\hat{\Gamma}_{p,k} = \sum_{m=0}^{L-1} \mathbf{R}_m (-1)^m \sum_{j=0}^{NL-1} P_{p,k;j}^{-1} \omega^j \sum_{\ell=1}^L \boldsymbol{\Omega}_\ell \mathbf{u}_\ell^{j-1-m}. \quad (80)$$

Unlike the \mathbf{u}_ℓ , the elements of \mathbf{P}^{-1} given in (72) are scalars multiplied by the unit operator, thus they commutes with all other operators. We denote the eigenvectors of the Hamiltonian \mathcal{H} by $|\{n_i\}\rangle = |n_1, \dots, n_k, \dots, n_L\rangle$ such that, as in (B.4.23),

$$\mathcal{H}|\{n_i\}\rangle = - \sum_{j=1}^L \omega^{n_j} r_j |\{n_i\}\rangle, \quad \text{and} \quad \mathbf{u}_\ell |\{n_i\}\rangle = r_\ell \omega^{n_\ell} |\{n_i\}\rangle. \quad (81)$$

We also rewrite (72) as

$$f_{p,k}(r_\ell \omega^{n_\ell}) = \sum_{j=0}^{NL-1} P_{p,k;j}^{-1} (r_\ell \omega^{n_\ell})^j = \delta_{k,\ell} \delta_{p,n_k}. \quad (82)$$

Consequently, (80) becomes

$$\begin{aligned} \widehat{\Gamma}_{p,k}|\{n_i\}\rangle &= \sum_{m=0}^{L-1} \sum_{\ell=1}^L (-1)^m \mathbf{R}_m \mathbf{\Omega}_\ell \mathbf{u}_\ell^{-1-m} |\{n_i\}\rangle \delta_{k,\ell} \delta_{n_k+1,p} \\ &= \delta_{n_k,p-1} \Omega_{p-1,k} \sum_{m=0}^{L-1} (-1)^m \mathbf{R}_m (\omega^{p-1} r_k)^{-1-m} |\{n_i\}\rangle, \end{aligned} \quad (83)$$

where

$$\mathbf{\Omega}_k |\{n_i\}\rangle = \Omega_{p-1,k} |\{n_i\}\rangle, \quad \Omega_{p-1,k} = 1 \Big/ \left[A_0 \prod_{i=1, i \neq k}^L (1 - \omega^{n_i-p+1} r_i / r_k) \right]. \quad (84)$$

Similarly, we may use the second formula in (36) to obtain

$$\langle \{n_i\} | \widehat{\Gamma}_{p,k} = \omega \delta_{n_k,p} \Omega_{p,k} \sum_{m=0}^{L-1} (-1)^m \langle \{n_i\} | \mathbf{R}_m (r_k \omega^p)^{-1-m}. \quad (85)$$

These results are in agreement with (B4.25) and surrounding text,

$$\widehat{\Gamma}_{p,k} |\{n_i\}\rangle = \widehat{\Gamma}_{p,k} |n_1, \dots, p-1, \dots, n_L\rangle = \Lambda_{p,k}(\{n_i\}) |n_1, \dots, p, \dots, n_L\rangle, \quad (86)$$

where $\Lambda_{p,k}$ depends on p and also on n_i for $i \neq k$. More precisely, $\Lambda_{p,k}$ is given, either by (83) or by (85), as

$$\Lambda_{p,k}(\{n_i\}) = (r_k \omega^{p-1})^{-1} \Omega_{p,k} \langle n_1, \dots, p, \dots, n_L | \mathbf{Y}(r_k \omega^p) | n_1, \dots, p-1, \dots, n_L \rangle \quad (87)$$

$$= (r_k \omega^{p-1})^{-1} \Omega_{p-1,k} \langle n_1, \dots, p, \dots, n_L | \mathbf{Y}(r_k \omega^{p-1}) | n_1, \dots, p-1, \dots, n_L \rangle. \quad (88)$$

in which

$$\mathbf{Y}(z) \equiv \sum_{m=0}^{L-1} (-1)^m \mathbf{R}_m z^{-m}. \quad (89)$$

From (86), we find that the $\widehat{\Gamma}_{p,k}$ behaves as cyclic raising operators. We shall now simplify the constant $\Lambda_{p,k}(\{n_i\})$.

5.4. Simplification of $\Lambda_{p,k}(\{n_i\})$

Since $\mathcal{H} = -\mathbf{A}_1/A_0$, we may use (49) to find

$$\begin{aligned} \mathbf{Y}(z) \mathcal{H} - \mathcal{H} \mathbf{Y}(z) &= (\omega - 1) \sum_{m=0}^{L-1} (-z)^{-m} \left[\mathbf{R}_0 \frac{\mathbf{A}_{m+1}}{A_0} - \mathbf{R}_{m+1} \right] \\ &= (1 - \omega) z \sum_{m=1}^L (-z)^{-m} \left[\mathbf{R}_0 \frac{\mathbf{A}_m}{A_0} - \mathbf{R}_m \right] = (1 - \omega) z \left[\mathbf{R}_0 \sum_{m=0}^L (-z)^{-m} \frac{\mathbf{A}_m}{A_0} - \mathbf{Y}(z) \right], \end{aligned} \quad (90)$$

where we have shifted the summation index by one, then used the fact that $\mathbf{R}_L = 0$ and finally extended the summation to include $m = 0$, as the zeroth term in the sum also vanishes identically. Now we can use (75) and (81) to rewrite the above equation as

$$\begin{aligned} \langle \{n'_i\} | \mathbf{Y}(z) | \{n_i\} \rangle & \left[z(1 - \omega) - \sum_{i=1}^L r_i(\omega^{n_i} - \omega^{n'_i}) \right] \\ & = z(1 - \omega) \langle \{n'_i\} | \mathbf{R}_0 | \{n_i\} \rangle \prod_{i=1}^L (1 - \omega^{n_i} r_i / z). \end{aligned} \quad (91)$$

If we let $z = \omega^{n_k} r_k$, the right-hand side is identically zero. Then, identically to what Baxter did, we find that for $\langle \{n'_i\} | \mathbf{Y}(\omega^{n_k} r_k) | \{n_i\} \rangle$ to be non-zero, we must have $n'_i = n_i$ for $i \neq k$ and $n'_k = n_k + 1$.

Therefore, for $z = \omega^{n_k+1} r_k$, with $n'_i = n_i$ for $i \neq k$, and $n'_k = n_k + 1$, we find

$$\langle \{n'_i\} | \mathbf{Y}(\omega^{n_k+1} r_k) | \{n_i\} \rangle = \langle \{n'_i\} | \mathbf{R}_0 | \{n_i\} \rangle \prod_{i=1, i \neq k}^L (1 - \omega^{n_i - n_k - 1} r_i / r_k). \quad (92)$$

Letting $n_k = p - 1$, and comparing the above equation with (87) and (84), we find

$$\Lambda_{p,k}(\{n_i\}) = A_0^{-1} (r_k \omega^{p-1})^{-1} \langle n_1, \dots, \overset{k}{p}, \dots, n_L | \mathbf{R}_0 | n_1, \dots, \overset{k}{p} - 1, \dots, n_L \rangle. \quad (93)$$

Now (54) can be used to show that (88) can be simplified to yield the identical result. From (24), (37) and (1), we find

$$\omega A_0^{-1} \mathbf{R}_0 = \mathbf{\Gamma}_1 = (\omega^{-1} - 1)^{-1} (\mathcal{H} \mathbf{\Gamma}_0 - \mathbf{\Gamma}_0 \mathcal{H}), \quad (94)$$

so that (93) can be even further simplified to

$$\Lambda_{p,k}(\{n_i\}) = \langle \{n'_i\} | \mathbf{\Gamma}_0 | \{n_i\} \rangle = \langle n_1, \dots, \overset{k}{p}, \dots, n_L | \mathbf{\Gamma}_0 | n_1, \dots, \overset{k}{p} - 1, \dots, n_L \rangle. \quad (95)$$

Thus to prove (B5.4), we need to prove

$$\begin{aligned} & (r_k \omega^{p-1} - r_\ell \omega^q) \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q}, \dots, n_L | \mathbf{\Gamma}_0 | n_1, \dots, \overset{k}{p} - 1, \dots, \overset{\ell}{q}, \dots, n_L \rangle \\ & \quad \langle n_1, \dots, \overset{k}{p} - 1, \dots, \overset{\ell}{q}, \dots, n_L | \mathbf{\Gamma}_0 | n_1, \dots, \overset{k}{p} - 1, \dots, \overset{\ell}{q} - 1, \dots, n_L \rangle + \\ & (r_\ell \omega^{q-1} - r_k \omega^p) \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q}, \dots, n_L | \mathbf{\Gamma}_0 | n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q} - 1, \dots, n_L \rangle \\ & \quad \langle n_1, \dots, \overset{k}{p}, \dots, \overset{\ell}{q} - 1, \dots, n_L | \mathbf{\Gamma}_0 | n_1, \dots, \overset{k}{p} - 1, \dots, \overset{\ell}{q} - 1, \dots, n_L \rangle = 0, \end{aligned} \quad (96)$$

which we have not yet succeeded in doing.

6. Summary

Let us now summarize the main steps in our proof of the conjectures of Baxter. As the first $\mathbf{\Gamma}$ in [24] is $\mathbf{\Gamma}_0 = \mathbf{Z}_1^{-1}$ in (B4.1), we split in (21) of section 2 the coefficients \mathbf{A}_ℓ in the expansion of $\tau_2(t)$ into hatted operators acting on sites 2 to L and operators (15) and (34) acting on site 1. Thus the hatted operators commute with $\mathbf{\Gamma}_0$, α_1^\pm and β_1^- . In subsection 4.1 we give the general formula (36) for $\mathbf{\Gamma}_j$, which we proved by induction. It was originally discovered calculating $\mathbf{\Gamma}_j$ for $j = 1, 2, 3$ using (22) and (23).

Conjecture (B4.3) is proved in subsection 4.2. We first express the coefficients s_j in terms of the \mathbf{A}_ℓ , see (59). We also rewrite (38) as (56), replacing the upper limits of the sums by L , as $\mathbf{A}_n = 0$ for $n > L$ and $\mathbf{q}_\ell = 0$ for $\ell < 0$. Likewise, we replace the upper limit of the summation in (36) by $L - 1$, as $\mathbf{R}_m = 0$ for $m > L - 1$. This allows us to interchange the summations in (62) and to show using (56) that (B4.3) holds.

In section 3, we proved that the coefficients of the expansion of $t\nu_1 = \mu_0$ in powers of t are equal using the commutation relations and (31). The proof of (B4.8) then follows by simple repeated application of (25).

In subsection 5.1, we show that the \mathbf{P} of (B4.16) diagonalizing the \mathbf{H} of (B4.10) is a Vandermonde matrix. Its inverse is therefore given by (72). In subsection 5.2, we show that the \mathbf{q}_ℓ defined in (38) are coefficients of the inverse of $\mathbf{A}(t)$, and thus have the alternative form (78). These equations are then used in subsection 5.3 to show that the $\hat{\Gamma}_{p,k}$ when acting on the eigenvectors of the Hamiltonian, behave as cyclic raising operators, see (86). The proportionality constant $\Lambda_{p,k}$ in (86) is simplified in subsection 5.4. We have not yet succeeded in proving (B5.4), but reduced it to a simpler form (96).

Since the $\tau_2(t)$ matrices considered here are most general, it may be interesting to see what these cyclic raising operators are in certain special cases, and to compare with Fendley's work [12, 13]. In particular, a proof of (B5.4) should also provide a proof of (111) in [14]. From (83) and (85), we see that the $\hat{\Gamma}_j$ are cyclic raising operators when acting on the right, and cyclic lowering operators when acting on the left. It should be interesting to find out what these operators do in the full integrable chiral Potts model.

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Appendix A. Yang–Baxter Equation

There are many different conventions for setting up the Yang–Baxter equation, which are slightly different, leading to different multiplicative factors and other changes. As an example, in our previous papers [27, 28], we have unfortunately used the convention of multiplying matrices from up to down, which causes $\mathbf{X} \rightarrow \mathbf{X}^{-1}$ as compared to Baxter's choice. Here we shall adopt Baxter's convention. For this reason, it may be good to provide some details of our setup used in the main text.

The products of four chiral Potts model weights [17] satisfy the Yang–Baxter

equation

$$\sum_{\alpha_2, \beta_2, \gamma_2} \bar{S}(rr'qq')_{\gamma_1, \beta_1}^{\gamma_2, \beta_2} S(pp'rr')_{\alpha_1, \gamma_2}^{\alpha_2, \gamma_3} S(pp'qq')_{\alpha_2, \beta_2}^{\alpha_3, \beta_3} = \sum_{\alpha_2, \beta_2, \gamma_2} S(pp'qq')_{\alpha_1, \beta_1}^{\alpha_2, \beta_2} S(pp'rr')_{\alpha_2, \gamma_1}^{\alpha_3, \gamma_2} \bar{S}(rr'qq')_{\gamma_2, \beta_2}^{\gamma_3, \beta_3}, \quad (\text{A.1})$$

as shown in figure A1. From the figure, we can also see that [17]

$$S(pp'qq')_{\alpha, \beta}^{\alpha', \beta'} = W_{p'q}(\alpha - \beta') \bar{W}_{p'q'}(\beta' - \alpha') \bar{W}_{pq}(\alpha - \beta) W_{pq'}(\beta - \alpha'). \quad (\text{A.2})$$

For the chiral Potts model, arrows must be drawn on the rapidity lines and also on the line pieces representing the Boltzmann weights. In our earlier papers, the matrix multiplications were done from up to down in order to have all S defined identically. Here, however, doing the multiplication in the other direction, we let

$$\bar{S}(rr'qq')_{\gamma, \beta}^{\gamma', \beta'} = S(rr'qq')_{\gamma', \beta}^{\gamma, \beta'} = W_{r'q}(\gamma' - \beta') \bar{W}_{r'q'}(\beta' - \gamma) \bar{W}_{rq}(\gamma' - \beta) W_{rq'}(\beta - \gamma). \quad (\text{A.3})$$

Indeed, from the figure we can see that we must interchange γ and γ' in order to be fully consistent with the four arrows on the S weights.

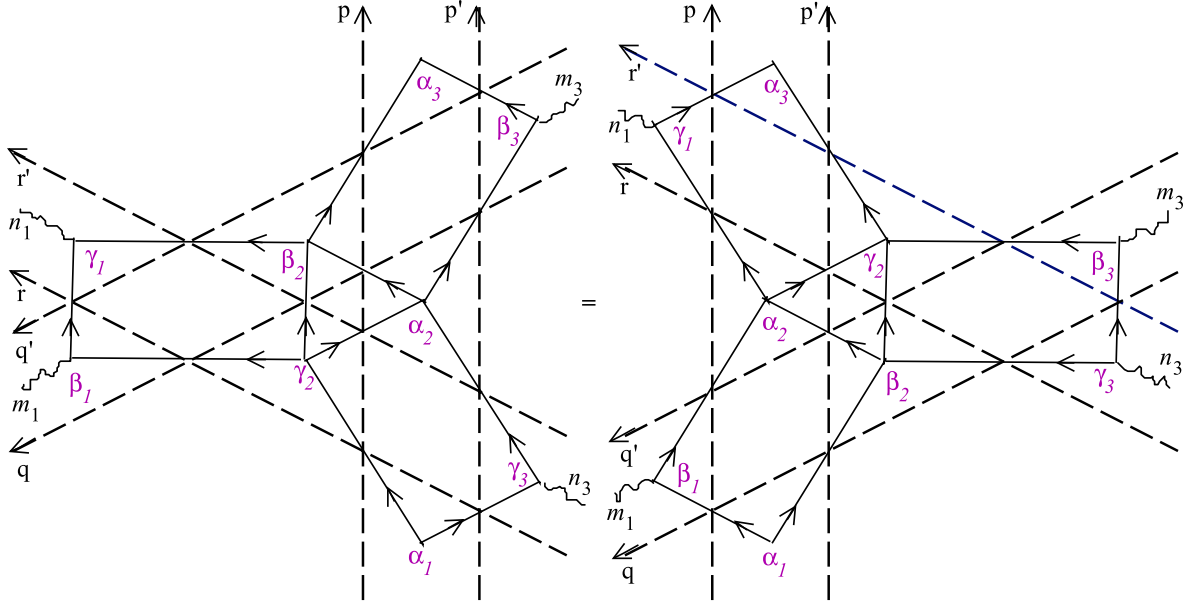


Figure A1. The Yang–Baxter equation for the chiral Potts model. The rapidity lines are represented by dashed oriented lines, the Boltzmann weights by oriented line pieces connecting pairs of spins.

Next we multiply both sides of (A.1) by $\omega^{-m_1\beta_1-n_1\gamma_1+m_3\beta_3+n_3\gamma_3}$, sum over $\beta_1, \beta_3, \gamma_1$ and γ_3 , and change one of the β_2 to β'_2 and one γ_2 to γ'_2 , while inserting

$$\delta_{\beta'_2\beta_2}\delta_{\gamma'_2\gamma_2} = N^{-2} \sum_{m_2, n_2} \omega^{m_2(\beta'_2-\beta_2)+n_2(\gamma'_2-\gamma_2)}$$

and summing over β'_2 and γ'_2 . Then, defining the Fourier transforms

$$S^{(pf)}(pp'qq')_{\alpha,m}^{\alpha',m'} = N^{-2} \sum_{\beta,\beta'} \omega^{-m\beta+m'\beta'} S(pp'qq')_{\alpha,\beta}^{\alpha',\beta'}, \quad (\text{A.4})$$

$$S^{(f)}(rr'qq')_{n,m}^{n',m'} = N^{-4} \sum_{\gamma,\gamma',\beta,\beta'} \omega^{-n\gamma-m\beta+n'\gamma'+m'\beta'} \bar{S}(rr'qq')_{\gamma,\beta}^{\gamma',\beta'}, \quad (\text{A.5})$$

the Yang–Baxter equation (A.1) becomes

$$\begin{aligned} \sum_{\alpha_2,m_2,n_2} S^{(f)}(rr'qq')_{n_1,m_1}^{n_2,m_2} S^{(pf)}(pp'rr')_{\alpha_1,n_2}^{\alpha_2,n_3} S^{(pf)}(pp'qq')_{\alpha_2,m_2}^{\alpha_3,m_3} \\ = \sum_{\alpha_2,m_2,n_2} S^{(pf)}(pp'qq')_{\alpha_1,m_1}^{\alpha_2,m_2} S^{(pf)}(pp'rr')_{\alpha_2,n_1}^{\alpha_3,n_2} S^{(f)}(rr'qq')_{n_2,m_2}^{n_3,m_3}. \end{aligned} \quad (\text{A.6})$$

As in [22, eq. (2.25)], we define

$$V_{pq'q}(\alpha, \alpha'; m) = N^{-1} \sum_{\beta} \omega^{m\beta} W_{pq}(\alpha - \beta) \bar{W}_{pq'}(\beta - \alpha'), \quad (\text{A.7})$$

so that (A.4) becomes

$$S^{(pf)}(p'pqq')_{\alpha,m}^{\alpha',m'} = V_{p'qq'}(\alpha, \alpha'; m') V_{pq'q}(-\alpha', -\alpha; m), \quad (\text{A.8})$$

whereas (A.5) can be rewritten as

$$S^{(f)}(rr'qq')_{n,m}^{n',m'} = N^{-2} \sum_{\gamma,\gamma'} \omega^{-n\gamma+n'\gamma'} V_{r'qq'}(\gamma', \gamma; m') V_{rq'q}(-\gamma, -\gamma'; m). \quad (\text{A.9})$$

It has been shown in [22] that if the rapidities q and q' are related by

$$(a_{q'}, b_{q'}, c_{q'}, d_{q'}) = (b_q, \omega^2 a_q, d_q, c_q), \quad (\text{A.10})$$

i.e. [22, eq. (2.28)] with $k = 0$ and $\ell = 2$, then $V_{p'qq'}(\alpha, \alpha'; m')$ is block-triangular. More precisely, when $0 \leq \alpha - \alpha' \leq 1$,

$$V_{p'qq'}(\alpha, \alpha'; m') = 0, \quad \text{for } 2 \leq m' \leq N - 1, \quad (\text{A.11})$$

while, for $0 \leq m' \leq 1$,

$$V_{p'qq'}(\alpha, \alpha'; m') = \Omega_{p'q} \omega^{m'\alpha'} (b_q/d_q)^{\alpha-\alpha'} (c_q/b_q)^{m'} F_{p'q}(\alpha - \alpha', m'). \quad (\text{A.12})$$

This is precisely [22, eq. (3.39a)] after using [22, eq. (3.21)] with $k = 0$ and $\ell = 2$ and $y_q = b_q/c_q$ [22, eq. (2.6)]. Under the same condition (A.10), $V_{pq'q}(-\alpha', -\alpha; m)$ is also found to be block-triangular, such that, for $0 \leq m \leq 1$,

$$V_{pq'q}(-\alpha', -\alpha; m) = 0, \quad \text{for } 2 \leq \alpha - \alpha' \leq N - 1, \quad (\text{A.13})$$

while it is non-vanishing for $0 \leq \alpha - \alpha' \leq 1$ and given by

$$V_{pq'q}(-\alpha', -\alpha; m) = \bar{\Omega}_{pq} \omega^{-m\alpha} (d_q/b_q)^{\alpha-\alpha'} (b_q/c_q)^m (-\omega t_q)^{\alpha-\alpha'-m} F_{pq}(\alpha - \alpha', m). \quad (\text{A.14})$$

This follows from [22, eq. (3.39b)] absorbing the factor $\bar{h}_{pq}^{(j)}$ into $\bar{\Omega}_{pq}$, while evaluating $\eta_{q,2,\alpha-\alpha'}/\eta_{q,2,m'}$ using [22, eq. (3.48)].

Consequently, we find the diagonal block of (A.8) for $0 \leq m', m \leq 1$ to be

$$S^{(pf)}(pp'qq')_{\alpha,m}^{\alpha',m'} = \bar{\Omega}_{pq} \Omega_{p'q} (b_q/c_q)^{m-m'} \mathcal{L}(pp'q)_{\alpha,m}^{\alpha',m'}, \quad (\text{A.15})$$

with \mathcal{L} given in (5), identifying $\mathcal{L}(pp'q)_{\alpha,m}^{\alpha',m'} = \mathcal{L}_j(m, m'; \alpha, \alpha')$ there. In $\bar{\Omega}_{pq}\Omega_{p'q}$ we have collected irrelevant factors that cancel out of the Yang–Baxter equation. Likewise if the two rapidities r and r' are also related by

$$(a_{r'}, b_{r'}, c_{r'}, d_{r'}) = (b_r, \omega^2 a_r, d_r, c_r), \quad (\text{A.16})$$

we have

$$S^{(pf)}(pp'rr')_{\alpha,n}^{\alpha',n'} = \bar{\Omega}_{pr}\Omega_{p'r}(b_r/c_r)^{n-n'}\mathcal{L}(pp'r)_{\alpha,n}^{\alpha',n'}. \quad (\text{A.17})$$

Consider now the Fourier transform (A.9). If q' and q are related by (A.10), we find from (A.13) that for $0 \leq m \leq 1$ that $V_{r'q}(-\gamma, -\gamma', m)$ is non-vanishing only when $\gamma' - \gamma = 0, 1$. Thus, if we change the sum over γ' to one over $\ell = \gamma' - \gamma$ and then sum over γ , we obtain

$$\begin{aligned} S^{(f)}(rr'qq')_{n,m}^{n',m'} &= \bar{\Omega}_{rq}\Omega_{r'q}(b_q/c_q)^{m-m'}N^{-2}\sum_{\gamma}\omega^{(-n+n'+m'-m)\gamma}\mathcal{R}(rr'q)_{n,m}^{n',m'} \\ &= \bar{\Omega}_{rq}\Omega_{r'q}(b_q/c_q)^{m-m'}N^{-1}\delta_{n'+m',n+m}\mathcal{R}(rr'q)_{n,m}^{n',m'}, \end{aligned} \quad (\text{A.18})$$

where

$$\mathcal{R}(rr'q)_{n,m}^{n',m'} = \sum_{\ell=0}^1 \omega^{(n'-m)\ell} F_{r'q}(\ell, m') F_{rq}(\ell, m) (-\omega t_q)^{\ell-m}. \quad (\text{A.19})$$

It is straightforward to show that when r' and r are also related by (A.16), $\mathcal{R}(rr'q)_{n,m}^{n',m'} = 0$ for $0 \leq n \leq 1$ and $2 \leq n' \leq N-1$, while for $0 \leq n, n' \leq 1$ it is given by

$$\delta_{n'+m',n+m}\mathcal{R}(rr'q)_{n,m}^{n',m'} = (b_r/c_r)^{m'-m}\mathcal{R}(rq)_{n,m}^{n',m'}, \quad (\text{A.20})$$

$$\mathcal{R}(rq)_{n,m}^{n',m'} = \delta_{n'+m',n+m} \left[\left(\frac{-t_q}{\omega t_r} \right)^{m'} - (-1)^{m'} \omega^{n-1} \left(\frac{t_q}{t_r} \right)^{1-m'} \right]. \quad (\text{A.21})$$

Here we used [22, eq. (3.48)] identifying $F_{pq}(\ell, m) = F_{pq}(2, \ell, m)$, which differs from (B.2.2) used to derive (6) by a normalization factor b_p . In particular, we have

$$\begin{aligned} \mathcal{R}(rq)_{0,0}^{0,0} &= \mathcal{R}(rq)_{1,1}^{1,1} = 1 - t_q/(\omega t_r), \\ \mathcal{R}(rq)_{1,0}^{1,0} &= \omega \mathcal{R}(rq)_{0,1}^{0,1} = 1 - t_q/t_r, \\ \mathcal{R}(rq)_{0,1}^{1,0} &= (t_r/t_q) \mathcal{R}(rq)_{1,0}^{0,1} = 1 - \omega^{-1}. \end{aligned} \quad (\text{A.22})$$

This shows that, when both relations in (A.10) and (A.16) hold, the Fourier transform of the product of four Boltzmann weights (A.9) reduces to the weights of a six-vertex model given as

$$S^{(f)}(rr'qq')_{n,m}^{n',m'} = \bar{\Omega}_{rq}\Omega_{r'q}(b_q/c_q)^{m-m'}(b_r/c_r)^{n-n'}N^{-1}\mathcal{R}(rq)_{n,m}^{n',m'}, \quad (\text{A.23})$$

for $0 \leq m, n, m', n' \leq 1$. Substituting (A.15), (A.17) and (A.23) into the Yang–Baxter equation (A.6), we find that many factors cancel out leaving us with

$$\begin{aligned} \sum_{\alpha_2, m_2, n_2} \mathcal{R}(rq)_{n_1, m_1}^{n_2, m_2} \mathcal{L}(pp'r)_{\alpha_1, n_2}^{\alpha_2, n_3} \mathcal{L}(pp'q)_{\alpha_2, m_2}^{\alpha_3, m_3} \\ = \sum_{\alpha_2, m_2, n_2} \mathcal{L}(pp'q)_{\alpha_1, m_1}^{\alpha_2, m_2} \mathcal{L}(pp'r)_{\alpha_2, n_1}^{\alpha_3, n_2} \mathcal{R}(rq)_{n_2, m_2}^{n_3, m_3}. \end{aligned} \quad (\text{A.24})$$

It is easily verified that this relation holds without any condition on the two sets of $\{a, b, c, d\}$ parameters making up the rapidities p and p' , unlike the Yang–Baxter equation for the chiral Potts model, for which the parameters have to satisfy [17, eq. 9] defining the chiral Potts curve. This observation has been made first by Baxter [25] in somewhat different notations.

Finally, it is obvious, that Yang–Baxter equation (A.24) also holds for so-called monodromy operators (9), replacing each \mathcal{L} by a product of \mathcal{L} -matrices sharing a horizontal rapidity line [29]. In particular, letting $n_1 = 0$, $m_1 = 1$, $n_3 = m_3 = 0$ in (A.24), we obtain (30). If we choose $n_1 = m_1 = 1$, $n_3 = m_3 = 0$, we find $\mathbf{C}(x)\mathbf{C}(y) = \mathbf{C}(y)\mathbf{C}(x)$, while using $n_1 = m_1 = n_3 = m_3 = 0$, we find $\mathbf{A}(x)\mathbf{A}(y) = \mathbf{A}(y)\mathbf{A}(x)$. Applying this to (20) we find the commutation relations

$$\hat{\mathbf{C}}_m \hat{\mathbf{C}}_n = \hat{\mathbf{C}}_n \hat{\mathbf{C}}_m, \quad \hat{\mathbf{A}}_m \hat{\mathbf{A}}_n = \hat{\mathbf{A}}_n \hat{\mathbf{A}}_m. \quad (\text{A.25})$$

Appendix B. Comparison with Fendley’s paper

Appendix B.1. Opening remarks

Before starting the comparison with [14], we must remark that we have to follow Baxter’s notations of [24], which used $\mathbf{\Gamma}_0 = \mathbf{Z}_1^{-1}$, rather than $\mathbf{\Gamma}_0 = \mathbf{Z}_1$ as used in [14, 15, 16]. This results in a spatial reflection of the way operators are multiplied. Therefore, we multiply operators in numerical order of site number, rather than Fendley’s (and Baxter’s earlier) anti-numerical order. Furthermore, in this appendix, equations in [14] will be denoted by prefacing F to their equation numbers.

Appendix B.2. Comparing transfer matrices

Following (B3.25), we set $a_j \equiv 0$ and $b_j \equiv 1$ in (6). As we now have

$$\begin{aligned} \alpha_j^+ &\equiv \mathbf{1}, & \beta_j^+ &= c_{2j-2} \mathbf{Z}_j^{-1}, & \gamma_j^+ &\equiv 0, \\ \alpha_j^- &= d_{2j-2} d_{2j-1} \mathbf{X}_j, & \beta_j^- &= c_{2j-1} \mathbf{Z}_j, & \gamma_j^- &= c_{2j-2} c_{2j-1} \mathbf{1}, \end{aligned} \quad (\text{B.1})$$

for the quantities defined in (12), (11) simplifies to

$$\mathcal{L}_j = \begin{bmatrix} \mathbf{1} & 0 \\ \beta_j^+ & 0 \end{bmatrix} - \omega t \begin{bmatrix} \alpha_j^- & \beta_j^- \\ 0 & \gamma_j^- \end{bmatrix}, \quad (\text{B.2})$$

with the special relationships

$$\begin{aligned} \alpha_j^- &= \mathbf{h}_{2j-1} \equiv d_{2j-2} d_{2j-1} \mathbf{X}_j, \\ \beta_j^- \beta_{j+1}^+ &= \mathbf{h}_{2j} \equiv c_{2j-1} c_{2j} \mathbf{Z}_j \mathbf{Z}_{j+1}^{-1}, & \beta_j^- \gamma_{j+1}^- &= \mathbf{h}_{2j} \beta_{j+1}^-. \end{aligned} \quad (\text{B.3})$$

Noting that $\tau_2(t)$ is defined in (9) and (10) as the 1-1 matrix element of $\prod_\ell \mathcal{L}_\ell$, it is then easily seen that the \mathbf{A}_ℓ also defined in (10) are expressed as sums of products of factors $\alpha_j^+ = \mathbf{1}$, $\alpha_j^- = \mathbf{h}_{2j-1}$ and

$$\beta_j^- \left(\prod_{i=j+1}^k \gamma_i^- \right) \beta_{k+1}^+ = \prod_{i=j}^k \mathbf{h}_{2i}. \quad (\text{B.4})$$

As we need precisely one of α_j^\pm , β_j^\pm , or γ_j^- for each site j , one can easily verify that we get the exclusion rule of (F41) that the subscripts of the \mathbf{h}_j must be at least two apart. More precisely, we find, in agreement with [14], that

$$\mathbf{A}_m = \sum_{i_1=1}^{2L-2m+1} \sum_{i_2=i_1+2}^{2L-2m+3} \cdots \sum_{i_m=i_{m-1}+2}^{2L-1} \prod_{j=1}^m \mathbf{h}_{i_j}, \quad (\text{B.5})$$

with the special cases

$$\mathbf{A}_0 = \mathbf{1}, \quad \mathbf{A}_L = \prod_{i=1}^L \mathbf{h}_{2i-1}. \quad (\text{B.6})$$

Therefore, for this special case, we have the following relation

$$T(-\omega t) = \tau_2(t), \quad (\text{B.7})$$

with $T(t)$ defined in (F50).

Appendix B.3. Comparing Hamiltonians

Next, as $\gamma_j^+ = 0$, only the term with $m = j + 1 < L$ survives within (14), so that now

$$\mathcal{H} = - \sum_{i=1}^{2L-1} \mathbf{h}_i, \quad (\text{B.8})$$

in agreement with (F34) and (F38) (up to a trivial minus sign) and with (B1.5) for this special case.

Appendix B.4. Comparing the eigenvalues of the Hamiltonian

The eigenvalues of the Hamiltonian are given in (81) in terms of $r_k \omega^{n_k}$. This has to be identified with $\epsilon_k \omega^{n_k}$ from the action of \mathcal{H} on the cyclic raising (shift) operator in (F95); one may also look at (F102) for $m = 1$. From (F48) and (B.7), we find

$$\sum_{j=0}^{N-1} t \frac{d}{dt} \ln[\tau_2(\omega^j t)] = \sum_{s=1} H^{(sN)} t^{sN}. \quad (\text{B.9})$$

Comparing (F63) and the equation above (F65) with (57) and (67) in the present paper, we can see that $u_k = \epsilon_k^N = r_k^N$, so that the eigenvalues indeed agree.

Appendix B.5. Relation between the inverses of Vandermonde matrices

In (72), we have expressed the elements $(P^{-1})_{ij}$ of the inverse of the Vandermonde matrix as coefficients of the polynomials $f_i(z)$. Identifying i in (72) with (p, k) and j with (q, ℓ) as is done in the above subsection (5.3), this can also be rewritten as

$$\begin{aligned} f_{p,k}(z) &= \prod_{\ell=1, \ell \neq k}^L \prod_{q=1}^N \frac{z - r_\ell \omega^q}{r_k \omega^p - r_\ell \omega^q} \prod_{q=1}^{N-1} \frac{z - r_k \omega^{p+q}}{r_k \omega^p (1 - \omega^q)} \\ &= \frac{1}{N} \prod_{\ell=1, \ell \neq k}^L \frac{z^N - r_\ell^N}{r_k^N - r_\ell^N} \left[\frac{(z/r_k)^N - 1}{z/(r_k \omega^p) - 1} \right]. \end{aligned} \quad (\text{B.10})$$

Obviously, we may express the product on the second line of (B.10) as the inverse of the Vandermonde matrix \mathcal{X} defined above (F100), and we may expand the part within the square brackets as a geometric series. Then, equating coefficients, we find

$$P_{i,\ell N+q}^{-1} = \frac{1}{N} (\mathcal{X}^{-1})_{k,\ell} (r_k \omega^p)^{-q}, \quad i \equiv (p, k). \quad (\text{B.11})$$

Consequently, we may rewrite (79) as

$$\hat{\Gamma}_{p,k} \equiv \sum_{j=0}^{NL-1} P_{ij}^{-1} \Gamma_j = \sum_{s=0}^{N-1} (r_k \omega^p)^{-s} \Phi_k^{(s)}(r_k^N), \quad \Phi_k^{(s)}(r_k^N) \equiv \frac{1}{N} \sum_{\ell=0}^{L-1} (\mathcal{X}^{-1})_{k,\ell} \Gamma_{\ell N+s}. \quad (\text{B.12})$$

It is easily verified that these functions $\Phi_k^{(s)}(r_k^N)$ satisfy the unnumbered relation below (F94) and the first unnumbered equation in section 5.3 of Fendley's paper and therefore

$$\hat{\Gamma}_{p,k} = \Psi_{\omega^p,k} \quad (\text{B.13})$$

with $\Psi_{\omega^p,k}$ defined in (F95).

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